Proc. Indian Acad. Sci. (Math. Sci.) Vol. 114, No. 1, February 2003, pp. 33–38.
Printed in India

Cobordism independence of Grassmann manifolds

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MS received 11 April 2003; revised 9 October 2003

Abstract. This note proves that, for $F = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, the bordism classes of all non-bounding Grassmannian manifolds $G_k(F^{n+k})$, with $k < n$ and having real dimension d, constitute a linearly independent set in the unoriented bordism group $\mathfrak{N}_d$ regarded as a $\mathbb{Z}_2$-vector space.

Keywords. Grassmannians; bordism; Stiefel–Whitney class.

1. Introduction

This paper is a continuation of the ongoing study of cobordism of Grassmann manifolds. Let F denote one of the division rings $\mathbb{R}$ of reals, $\mathbb{C}$ of complex numbers, or $\mathbb{H}$ of quaternions. Let $t = \dim_{\mathbb{R}} F$. Then the Grassmannian manifold $G_k(F^{n+k})$ is defined to be the set of all k-dimensional (left) subspaces of F^{n+k}. $G_k(F^{n+k})$ is a closed manifold of real dimension nkt. Using the orthogonal complement of a subspace one identifies $G_k(F^{n+k})$ with $G_n(F^{n+k})$.

In [8], Sankaran has proved that, for $F = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, the Grassmannian manifold $G_k(F^{n+k})$ bounds if and only if $\nu(n+k) > \nu(k)$, where, given a positive integer m, $\nu(m)$ denotes the largest integer such that $2^{\nu(m)}$ divides m.

Given a positive integer d, let $\mathscr{G}(d)$ denote the set of bordism classes of all non-bounding Grassmannian manifolds $G_k(F^{n+k})$ having real dimension d such that $k < n$. The restriction $k < n$ is imposed because $G_k(F^{n+k}) \approx G_n(F^{n+k})$ and, for $k = n$, $G_k(F^{n+k})$ bounds. Thus, $\mathscr{G}(d) = \{[G_k(F^{n+k})] \in \mathfrak{N}_* \mid nkt = d, k < n, \text{ and } \nu(n+k) \leq \nu(k)\} \subset \mathfrak{N}_d$.

The purpose of this paper is to prove the following:

Theorem 1.1. *$\mathscr{G}(d)$ is a linearly independent set in the $\mathbb{Z}_2$-vector space $\mathfrak{N}_d$.*

Similar results for Dold and Milnor manifolds can be found in [6] and [1] respectively.

2. The real Grassmannians — a Brief review

The real Grassmannian manifold $G_k(\mathbb{R}^{n+k})$ is an nk-dimensional closed manifold of k-planes in $\mathbb{R}^{n+k}$. It is well-known (see [3]) that the mod-2 cohomology of $G_k(\mathbb{R}^{n+k})$ is given by

$$H^*(G_k(\mathbb{R}^{n+k});\, \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, w_2, \ldots, w_k, \overline{w}_1, \overline{w}_2, \ldots, \overline{w}_n]/\{w.\overline{w} = 1\},$$

where $w = 1 + w_1 + w_2 + \cdots + w_k$ and $\overline{w} = 1 + \overline{w}_1 + \overline{w}_2 + \cdots + \overline{w}_n$ are the total Stiefel–Whitney classes of the universal k-plane bundle γ_k and the corresponding complementary bundle $\gamma_k^{\perp}$, both over $G_k(\mathbb{R}^{n+k})$, respectively.

For computational convenience in this cohomology one uses the flag manifold $\mathrm{Flag}(\mathbb{R}^{n+k})$ consisting of all ordered $(n+k)$-tuples $(V_1, V_2, \ldots, V_{n+k})$ of mutually orthogonal one-dimensional subspaces of $\mathbb{R}^{n+k}$ with respect to the 'standard' inner product on $\mathbb{R}^{n+k}$. It is standard (see [4]) that the mod-2 cohomology of $\mathrm{Flag}(\mathbb{R}^{n+k})$ is given by

$$H^*(\mathrm{Flag}(\mathbb{R}^{n+k});\ \mathbb{Z}_2) \cong \mathbb{Z}_2[e_1, e_2, \ldots, e_{n+k}] \Big/ \left\{ \prod_{i=1}^{n+k} (1 + e_i) = 1 \right\},$$

where $e_1, e_2, \ldots, e_{n+k}$ are one-dimensional classes. In fact each e_i is the first Stiefel–Whitney class of the line bundle λ_i over $\mathrm{Flag}(\mathbb{R}^{n+k})$ whose total space consists of pairs, a flag $(V_1, V_2, \ldots, V_{n+k})$ and a vector in V_i.

There is a map $\pi_{n+k} : \mathrm{Flag}(\mathbb{R}^{n+k}) \longrightarrow G_k(\mathbb{R}^{n+k})$ which assigns to $(V_1, V_2, \ldots, V_{n+k})$, the k-dimensional subspace $V_1 \oplus V_2 \oplus \cdots \oplus V_k$. In the cohomology, $\pi_{n+k}^* : H^*(G_k(\mathbb{R}^{n+k});\ \mathbb{Z}_2) \longrightarrow H^*(\mathrm{Flag}(\mathbb{R}^{n+k});\ \mathbb{Z}_2)$ is injective and is described by

$$\pi_{n+k}^*(w) = \prod_{i=1}^{k} (1 + e_i), \quad \pi_{n+k}^*(\overline{w}) = \prod_{i=k+1}^{n+k} (1 + e_i).$$

In [9], Stong has observed, among others, the following facts:

Fact 2.1. The value of the class $u \in H^*(G_k(\mathbb{R}^{n+k});\ \mathbb{Z}_2)$ on the fundamental class of $G_k(\mathbb{R}^{n+k})$ is the same as the value of

$$\pi_{n+k}^*(u) e_1^{k-1} e_2^{k-2} \ldots e_{k-1} e_{k+1}^{n-1} e_{k+2}^{n-2} \ldots e_{n+k-1}$$

on the fundamental class of $\mathrm{Flag}(\mathbb{R}^{n+k})$.

Fact 2.2. In $H^*(\mathrm{Flag}(\mathbb{R}^{n+k});\ \mathbb{Z}_2)$ one has

$$e_{i_1}^{n+k-(r-1)} e_{i_2}^{n+k-(r-1)} \ldots e_{i_{r-1}}^{n+k-2} e_{i_r}^{n+k-1} \ = \ 0$$

if $1 \leq r \leq n+k$ and the set $\{i_1, i_2, \ldots, i_r\} \subset \{1, 2, \ldots, n+k\}$. In particular $e_i^{n+k} \ = \ 0$ for each i, $1 \leq i \leq n+k$.

Fact 2.3. In the top dimensional cohomology of $\mathrm{Flag}(\mathbb{R}^{n+k})$, a monomial $e_1^{i_1} e_2^{i_2} \ldots e_{n+k}^{i_{n+k}}$ represents the non-zero class if and only if the set $\{i_1, i_2, \ldots, i_{n+k}\} = \{0, 1, \ldots, n+k-1\}$.

The tangent bundle τ over $G_k(\mathbb{R}^{n+k})$ is given (see [5]) by

$$\tau \oplus \gamma_k \otimes \gamma_k \cong (n+k)\gamma_k.$$

In particular, the total Stifel–Whitney class $W(G_k(\mathbb{R}^{n+k}))$ of the tangent bundle over $G_k(\mathbb{R}^{n+k})$ maps under π_{n+k}^* to

$$\prod_{1 \leq i \leq k} (1 + e_i)^{n+k} \cdot \prod_{1 \leq i < j \leq k} (1 + e_i + e_j)^{-2}.$$

Choosing a positive integer α such that $2^{\alpha} \geq n+k$, we have, using Fact 2.2,

$$\pi_{n+k}^{*}(W(G_k(\mathbb{R}^{n+k}))) = \prod_{1\leq i\leq k}(1+e_i)^{n+k} \cdot \prod_{1\leq i<j\leq k}(1+e_i+e_j)^{2^{\alpha}-2}.$$

Thus, the mth Stiefel–Whitney class $W_m = W_m(G_k(\mathbb{R}^{n+k}))$ maps under π_{n+k}^{*} to the mth elementary symmetric polynomial in e_i, $1 \leq i \leq k$, each with multiplicity $n+k$, and e_i+e_j, $1 \leq i < j \leq k$, each with multiplicity $2^{\alpha}-2$. Therefore, if $S_p(\sigma_1, \sigma_2, \ldots, \sigma_p)$ denotes the expression of the power sum $\sum_{m=1}^{q} y_m^p$ as a polynomial in elementary symmetric polynomials σ_m's in q 'unknowns' $y_1, y_2, \ldots, y_q$, $q \geq p$, we have (see [8])

$$S_p(\pi_{n+k}^{*}(W_1), \pi_{n+k}^{*}(W_2), \ldots, \pi_{n+k}^{*}(W_p)) = \sum_{1\leq i\leq k}(n+k)e_i^p.$$

Thus we have a polynomial

$$S_p(G_k(\mathbb{R}^{n+k})) = S_p(W_1, W_2, \ldots, W_p) \in H^p(G_k(\mathbb{R}^{n+k});\ \mathbb{Z}_2)$$

of Stiefel–Whitney classes of $G_k(\mathbb{R}^{n+k})$ such that

$$\pi_{n+k}^{*}(S_p(G_k(\mathbb{R}^{n+k}))) = \begin{cases} \sum\limits_{1\leq i\leq k} e_i^p, & \text{if } n+k \text{ is odd and } p < n+k \\ 0, & \text{otherwise.} \end{cases} \tag{2.4}$$

3. Proof of Theorem 1.1

It is shown in [2] that

$$[G_{2k}(\mathbb{R}^{2n+2k})] = [G_k(\mathbb{R}^{n+k})]^4 \quad \text{in } \mathfrak{N}_{4nk}.$$

From this, we have, in particular,

$$[G_k(F^{n+k})] = [G_k(\mathbb{R}^{n+k})]^t \quad \text{in } \mathfrak{N}_{nkt}.$$

For this one has to simply observe that the mod-2 cohomology of the $\mathbb{F}$-Grassmannian is isomorphic as ring to that of the corresponding real Grassmannian by an obvious isomorphism that multiplies the degree by t. On the other hand, since $\mathfrak{N}_*$ is a polynomial ring over the field $\mathbb{Z}_2$, we have the following:

Remark 3.1. A set $\{[M_1], [M_2], \ldots, [M_m]\}$ is linearly independent in $\mathfrak{N}_d$ if and only if the set $\{[M_1]^{2^{\beta}}, [M_2]^{2^{\beta}}, \ldots, [M_m]^{2^{\beta}}\}$ is linearly independent in $\mathfrak{N}_{d.2^{\beta}}$, $\beta \geq 0$.

Therefore, noting that $t = 1, 2$, or 4, it is enough to prove Theorem 1.1 for real Grassmannians only. Thus, from now onwards, we shall take

$$\mathscr{G}(d) = \{[G_k(\mathbb{R}^{n+k})] \mid nk = d,\ k < n, \text{ and } \nu(n+k) \leq \nu(k)\}.$$

If $G_k(\mathbb{R}^{n+k})$ is an odd-dimensional real Grassmannian manifold then both n and k must be odd, and so $\nu(n+k) > \nu(k)$. This means that $G_k(\mathbb{R}^{n+k})$ bounds and so it follows that $\mathscr{G}(d) = \emptyset$ if d is odd. Therefore we assume that d is even.

Lemma 3.2. *In* $H^*(\mathrm{Flag}(\mathbb{R}^{n+k});\ \mathbb{Z}_2)$ *one has, for* $1 \le j \le k$,

$$\left(\sum_{1\le i\le k} e_i^{n+k-(2j-1)}\right)\cdot e_1^{k-1}e_2^{k-2}\dots e_{k-j}^{j}\cdot e_{k-(j-1)}^{j-1}\cdot e_{k-(j-2)}^{n+k-(j-1)}\dots e_k^{n+k-1}$$
$$= e_1^{k-1}e_2^{k-2}\dots e_{k-j}^{j}\cdot e_{k-(j-1)}^{n+k-j}\cdot e_{k-(j-2)}^{n+k-(j-1)}\dots e_k^{n+k-1}.$$

Proof. Note that

(a) if $i \neq k-(j-1)$ then the exponent of e_i in the product

$$e_1^{k-1}e_2^{k-2}\dots e_{k-j}^{j}\cdot e_{k-(j-1)}^{j-1}\cdot e_{k-(j-2)}^{n+k-(j-1)}\dots e_k^{n+k-1}$$

is greater than or equal to j, and
(b) $\{n+k-(2j-1)\}+j = n+k-(j-1)$.

Therefore, invoking Fact 2.2, the lemma follows.

PROPOSITION 3.3.

Let $\mathscr{O}(d) = \{[G_k(\mathbb{R}^{n+k})] \in \mathscr{G}(d) \ \mid\ n+k \text{ is odd }\}$. *Then* $\mathscr{O}(d)$ *is linearly independent in* $\mathfrak{N}_d$.

Proof. Arrange the members of $\mathscr{O}(d)$ in descending order of the values of $n+k$, so that

$$\mathscr{O}(d) = \{[G_{k_1}(\mathbb{R}^{n_1+k_1})],\ [G_{k_2}(\mathbb{R}^{n_2+k_2})],\ \dots,\ [G_{k_s}(\mathbb{R}^{n_s+k_s})]\},$$

where $n_1+k_1 > n_2+k_2 > \dots > n_s+k_s$. Note that $n_1 = d$ and $k_1 = 1$.

For a d-dimensional Grassmannian manifold $G_k(\mathbb{R}^{n+k})$, consider the polynomials

$$f_\ell(G_k(\mathbb{R}^{n+k})) \ = \prod_{1\le j\le k_\ell} S_{n_\ell+k_\ell-(2j-1)}(G_k(\mathbb{R}^{n+k})) \ \in H^d(G_k(\mathbb{R}^{n+k});\mathbb{Z}_2)$$

of Stiefel–Whitney classes of $G_k(\mathbb{R}^{n+k})$, where $1 \le \ell \le s$. Then, for each ℓ, $1 \le \ell \le s$, we have, using (2.4),

$$\pi^*_{n_\ell+k_\ell}(f_\ell(G_{k_\ell}(\mathbb{R}^{n_\ell+k_\ell})))e_1^{k_\ell-1}e_2^{k_\ell-2}\dots e_{k_\ell-1}e_{k_\ell+1}^{n_\ell-1}e_{k_\ell+2}^{n_\ell-2}\dots e_{n_\ell+k_\ell-1}$$
$$= \left(\prod_{1\le j\le k_\ell}\left(\sum_{1\le i\le k_\ell} e_i^{n_\ell+k_\ell-(2j-1)}\right)\right)e_1^{k_\ell-1}e_2^{k_\ell-2}\dots e_{k_\ell-1}e_{k_\ell+1}^{n_\ell-1}e_{k_\ell+2}^{n_\ell-2}$$
$$\dots e_{n_\ell+k_\ell-1}$$
$$= e_1^{n_\ell}e_2^{n_\ell+1}\dots e_{k_\ell}^{n_\ell+k_\ell-1}e_{k_\ell+1}^{n_\ell-1}e_{k_\ell+2}^{n_\ell-2}\dots e_{n_\ell+k_\ell-1},$$

applying Lemma 3.2 repeatedly for successive values of j.

Thus, in view of Facts 2.1 and 2.3, the Stiefel–Whitney number

$$\langle f_\ell(G_{k_\ell}(\mathbb{R}^{n_\ell+k_\ell})),[G_{k_\ell}(\mathbb{R}^{n_\ell+k_\ell})]\rangle \ \neq\ 0$$

for each ℓ, $1 \le \ell \le s$. On the other hand, using (2.4), it is clear that

$$\langle f_\ell(G_{k_h}(\mathbb{R}^{n_h+k_h})),[G_{k_h}(\mathbb{R}^{n_h+k_h})]\rangle \ =\ 0$$

for each $h > \ell$, since $n_\ell + k_\ell - 1 \ \geq \ n_h + k_h$. Therefore, it follows that the $s \times s$ matrix

$$[\langle f_\ell(G_{k_h}(\mathbb{R}^{n_h+k_h})), [G_{k_h}(\mathbb{R}^{n_h+k_h})]\rangle]_{1\leq \ell\leq s,\, 1\leq h\leq s}$$

is non-singular; being lower triangular with 1's in the diagonal. This completes the proof.

Now we shall complete the proof of Theorem 1.1 using induction on d. First note that

$$\mathscr{G}(2) = \{[G_1(\mathbb{R}^{2+1})]\} = \{[\mathbb{R}P^2]\},$$
$$\mathscr{G}(4) = \{[G_1(\mathbb{R}^{4+1})]\} = \{[\mathbb{R}P^4]\},$$

and so both are linearly independent in $\mathfrak{N}_2$, $\mathfrak{N}_4$ respectively. Assume that the theorem holds for all dimensions less than d.

We have $\mathscr{G}(d) \ = \ \mathscr{E}(d) \bigcup \mathscr{O}(d)$, where

$$\mathscr{E}(d) \ = \ \{[G_k(\mathbb{R}^{n+k})] \in \mathscr{G}(d) | n+k \ \text{ is even}\}$$

and

$$\mathscr{O}(d) \ = \ \{[G_k(\mathbb{R}^{n+k})] \in \mathscr{G}(d) | n+k \ \text{ is odd}\}.$$

Observe that if $[G_k(\mathbb{R}^{n+k})] \in \mathscr{E}(d)$ then both n and k are even with $\nu(k) \neq \nu(n)$. On the other hand, $[G_2(\mathbb{R}^{\frac{d}{2}+2})] \ \in \ \mathscr{E}(d)$ if $d \equiv 0 \pmod 8$. Thus, $\mathscr{E}(d) \neq \emptyset$ if and only if $d \equiv 0$ $\pmod 8$.

In view of Proposition 3.3, we may assume without any loss that $\mathscr{E}(d) \ \neq \ \emptyset$. Then, by the above observation and by Theorem 2.2 of [8] every member of $\mathscr{E}(d)$ is of the form $[G_{\frac{k}{2}}(\mathbb{R}^{\frac{n}{2}+\frac{k}{2}})]^4$, where $[G_{\frac{k}{2}}(\mathbb{R}^{\frac{n}{2}+\frac{k}{2}})] \ \in \ \mathscr{G}(\frac{d}{4})$. By induction hypothesis, $\mathscr{G}(\frac{d}{4})$ is linearly independent in $\mathfrak{N}_{\frac{d}{4}}$.

So, by Remark 3.1,

$$\mathscr{E}(d) \text{ is linearly independent in } \mathfrak{N}_d. \tag{3.4}$$

Again note that if $[G_k(\mathbb{R}^{n+k})] \ \in \ \mathscr{E}(d)$, then, by (2.4), the polynomial $S_p(G_k(\mathbb{R}^{n+k})) = 0$, $\forall \ p \geq 1$. So, for each of the polynomials f_ℓ, $\ 1 \leq \ell \leq s$, considered in Proposition 3.3, we have

$$\langle f_\ell(G_k(\mathbb{R}^{n+k})), [G_k(\mathbb{R}^{n+k})]\rangle \ = \ 0.$$

Therefore, writing

$$\mathscr{E}(d) = \{[G_{k_{s+1}}(\mathbb{R}^{n_{s+1}+k_{s+1}})], [G_{k_{s+2}}(\mathbb{R}^{n_{s+2}+k_{s+2}})], \ldots, [G_{k_{s+q}}(\mathbb{R}^{n_{s+q}+k_{s+q}})]\},$$

where $n_{s+1} + k_{s+1} > n_{s+2} + k_{s+2} > \cdots > n_{s+q} + k_{s+q}$, we see that the $s \times (s+q)$ matrix

$$[\langle f_\ell(G_{k_h}(\mathbb{R}^{n_h+k_h})), [G_{k_h}(\mathbb{R}^{n_h+k_h})]\rangle]_{1\leq \ell\leq s,\, 1\leq h\leq s+q}$$

is of the form

$$\begin{array}{c}
\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\star & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\star & \star & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
- & - & - & \cdots & - & - & - & - & \cdots & - \\
- & - & - & \cdots & - & - & - & - & \cdots & - \\
\star & \star & \star & \cdots & 1 & 0 & 0 & 0 & \cdots & 0
\end{bmatrix} \\
\qquad\qquad \mathscr{O}(d) \qquad\qquad\qquad\qquad \mathscr{E}(d)
\end{array}. \tag{3.5}$$

Thus, no non-trivial linear combination of members of $\mathscr{O}(d)$ can be expressed as a linear combination of the members of $\mathscr{E}(d)$. This, together with (3.4) and Proposition 3.3, proves that the set $\mathscr{G}(d) = \mathscr{E}(d) \bigcup \mathscr{O}(d)$ is linearly independent in $\mathfrak{N}_d$. Hence, by induction, Theorem 1.1 is completely proved.

Remark 3.6. Using the decomposition of the members of $\mathscr{E}(d)$, and the polynomials f_ℓ, in the lower dimensions together with the *doubling homomorphism* defined by Milnor [7], one can obtain a set of polynomials of Stiefel–Whitney classes which yield, as in Proposition 3.3, a lower triangular matrix for $\mathscr{E}(d)$ with 1's in the diagonal. Thus using (3.5) we have a lower triangular matrix, with 1's in the diagonal, for the whole set $\mathscr{G}(d)$.

Acknowledgement

Part of this work was done under a DST project

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